

DIFFERENTIAL POSETS AND SMITH NORMAL FORMS

ALEXANDER MILLER AND VICTOR REINER

ABSTRACT. We conjecture a strong property for the up and down maps U and D in an r -differential poset: $DU + tI$ and $UD + tI$ have Smith normal forms over $\mathbb{Z}[t]$. In particular, this would determine the integral structure of the maps U, D, UD, DU , including their ranks in any characteristic.

As evidence, we prove the conjecture for the Young-Fibonacci lattice \mathbf{YF} studied by Okada and its r -differential generalizations $Z(r)$, as well as verifying many of its consequences for Young's lattice \mathbf{Y} and the r -differential Cartesian products Y^r .

1. INTRODUCTION AND THE MAIN CONJECTURE

The focus of this paper is Conjecture 1.1 below. We provide here some background and motivation before stating it.

For a positive integer r , an r -differential poset is a graded poset P with a minimum element, having all intervals and all rank cardinalities finite, satisfying two crucial extra conditions:

- (D1) If an element of P covers m others, then it will be covered by $m + r$ others.
- (D2) If two elements of P have exactly m elements that they both cover, then there will be exactly m elements that cover them both.

Favorite examples include the 1-differential poset *Young's lattice* \mathbf{Y} of all number partitions ordered by inclusion of their Ferrers diagrams, the r -differential Cartesian product order Y^r , and the Fibonacci posets $Z(r)$ discussed in below. See Stanley [19, 20] for more on differential posets, and Fomin [4, 5] for the more general notion of *dual graded graphs*.

There is an important rephrasing of the properties (D1, D2) above. Let P_n denote the n^{th} rank of P , say with rank size $p_n := |P_n|$. For any ring R , consider the free R -module R^{p_n} having basis indexed by the elements of P_n , and define the *up* and *down* maps

$$\begin{aligned} U_n(= D_{n+1}^t) : R^{p_n} &\rightarrow R^{p_{n+1}} \\ D_n(= U_{n-1}^t) : R^{p_n} &\rightarrow R^{p_{n-1}} \end{aligned}$$

in which a basis element is sent by U_n (resp. D_n) to the sum, with coefficient 1, of all elements that cover (resp. are covered by) it. Using the abbreviated notations

$$\begin{aligned} UD_n &:= U_{n-1}D_n \\ DU_n &:= D_{n+1}U_n \end{aligned}$$

Key words and phrases. invariant factors, Smith normal form, differential poset, dual graded graphs.

The authors thank Janvier Nzeutchap, John Shareshian, and Richard Stanley for helpful conversations, and they particularly thank Patrick Byrnes for helpful computations as well as conversations.

the conditions (D1, D2) can be rephrased as saying that, for each $n \geq 1$,

$$DU_n - UD_n = rI$$

or equivalently, that

$$(1.1) \quad DU_n = UD_n + rI.$$

This turns out to imply that, working over rings R of characteristic zero, U_n, D_n are of maximal rank, via a simple calculation of the eigenvalues and multiplicities for the nonnegative definite operators DU_n, UD_n ; see Proposition 2.1 below. The main conjecture of this paper, Conjecture 1.1 below, gives further information about the maps DU_n, UD_n (as well as U_n, D_n themselves) which is independent of the ring R .

For a ring R and an $m \times n$ matrix A in $R^{m \times n}$ say that A

- *is in Smith form* if it is diagonal in the sense that $a_{ij} = 0$ for $i \neq j$, and its diagonal entries $s_i := a_{ii}$ have s_i dividing s_{i+1} in R for each i .
- *has a Smith form over R* if it can be brought into Smith form by change-of-bases in the free modules R^m, R^n , that is, if there exist matrices P in $GL_m(R)$ and Q in $GL_n(R)$ for which PAQ is in Smith form.
- *has a parametrized Smith form for $A + tI$* if A is square and the matrix $A + tI$ in $R[t]^{n \times n}$ has a Smith form over $R[t]$.

We will mainly be interested in the case where $R = \mathbb{Z}$. In this situation, since \mathbb{Z} is a principal ideal domain (PID), having a Smith form for a matrix A over R is guaranteed, but most square matrices A over R will *not* have a parametrized Smith form for $A + tI$, as $\mathbb{Z}[t]$ is not a PID. Having such a parametrized Smith form leads to very strong consequences, including a prediction for the Smith forms over \mathbb{Z} for all specializations $A + kI$; see the Appendix (Section 8) below.

Conjecture 1.1. *In an r -differential poset, for every $n \geq 0$, DU_n in $\mathbb{Z}^{p_n \times p_n}$ has a parametrized Smith form for $DU_n + tI$ over $\mathbb{Z}[t]$.*

Note that due to equation (1.1), it would be equivalent to conjecture that UD_n when $n \geq 1$ has a parametrized Smith form for $UD_n + tI$, as the ring automorphism of $\mathbb{Z}[t]$ that maps $t \mapsto t + r$ would convert one Smith form to the other.

The paper is structured as follows.

Section 2 discusses two conjectures (Conjectures 2.3 and 2.4) on r -differential posets that are closely related to Conjecture 1.1.

Section 3 discusses some computer experiments checking consequences of Conjecture 1.1.

Section 4 discusses two standard constructions for producing r -differential posets, and their behavior with respect to the conjectures.

Section 5 proves all of the conjectures for the Young-Fibonacci poset \mathbf{YF} and its r -differential generalizations $Z(r)$.

Section 6 discusses Young's lattice \mathbf{Y} and the r -fold Cartesian products \mathbf{Y}^r . Although we have not been able to prove Conjecture 1.1 in these cases, some of its predictions are verified here.

Section 7 discusses some remarks and questions regarding attempts to generalize parts of the conjecture to the *dual graded graphs* of Fomin [4].

The appendix Section 8 discusses general facts about Smith forms over rings which are not necessarily PID's, and the consequences of having a parametrized Smith form for $A + tI$.

2. TWO RELATED CONJECTURES

In all of the conjectures in this section, P is an r -differential poset, with rank sizes p_n , up maps U_n and down maps $D_{n+1} = U_n^t$ for $n \geq 0$.

2.1. On the rank sizes. It turns out that in the presence of a certain numerical condition on the rank sizes p_n (Conjecture 2.3 below), the main Conjecture 1.1 implies and is closely related to a weaker conjecture (Conjecture 2.4 below).

We recall here an observation of Stanley concerning how the rank sizes p_n in a differential poset determine the spectra of the operators DU_n and UD_n .

Proposition 2.1. [19, §4] *Let P be an r -differential poset. Then*

$$\det(DU_n + tI) = \prod_{i=0}^n (t + r(i+1))^{\Delta p_{n-i}}$$

$$\det(UD_n + tI) = \prod_{i=0}^n (t + ri)^{\Delta p_{n-i}}$$

where $\Delta p_n := p_n - p_{n-1}$.

In particular, as DU has all positive integer eigenvalues,

- DU_n is nonsingular if R has characteristic zero,
- the up maps U_n are injective if R has characteristic zero,
- the down maps D_n are surjective if R has characteristic zero, and
- the rank sizes weakly increase: $p_0 \leq p_1 \leq p_2 \leq \dots$.

Since one knows the eigenvalues of UD_n , and they lie in \mathbb{Z} , Proposition 8.4 below shows that whenever Conjecture 1.1 holds, the Smith normal form entries $s_1(t), \dots, s_{p_n}(t)$ for $DU_n + tI$ take the following explicit form in terms of the rank sizes p_n :

$$(2.1) \quad s_{p_{n+1}-j}(t) = \prod_{\substack{i=0 \\ \Delta p_{n-i} \geq j}}^n (t + r(i+1))$$

for $j = 1, 2, \dots, p_n$, where an empty product is taken to be 1. In particular, this predicts for any chosen element k in the ring R the structure of the kernel or cokernel of $DU_n + kI$ or $UD_n + kI$, and if R is a field, will predict the ranks of $DU_n + kI, UD_n + kI$ in terms of the characteristic of the field.

In light of Proposition 2.1, an affirmative answer to the following question of Stanley on the *strict* increase of the p_n , or positivity of the first differences Δp_n , would determine the *support* of the spectra of the operators DU_n and UD_n , and hence determine the last Smith entry $s_{p_n}(t)$ of $DU_n + tI$.

Question 2.2. (Stanley [19, §6, end of Problem 6].)

In an r -differential poset P , do the rank sizes p_n satisfy

$$(r=)p_1 < p_2 < p_3 < \dots?$$

We make the following conjecture on the rank sizes.

Conjecture 2.3. *For every $n \geq 1$ and $1 \leq j \leq n$, one has*

$$\Delta p_n \geq \Delta p_{n-j-\delta_{r,1}} = \begin{cases} \Delta p_{n-j-1} & \text{if } P \text{ is 1-differential,} \\ \Delta p_{n-j} & \text{if } P \text{ is } r\text{-differential for } r \geq 2. \end{cases}$$

In other words, for $r \geq 2$, one has

$$(2.2) \quad \Delta p_1 \leq \Delta p_2 \leq \cdots$$

and for $r = 1$ one has

$$\Delta p_1, \Delta p_2, \dots, \Delta p_{n-3}, \Delta p_{n-2} \leq \Delta p_n.$$

Conjecture 2.3 has the following useful rephrasing, which is equivalent via Proposition 2.1: the zero eigenvalue of UD_n achieves the largest multiplicity among all of its *non-unit* eigenvalues.

We remark that for many known examples of r -differential posets P (but not all; see Section 3 below), the stronger condition (2.2) also holds for $r = 1$. In other words, the *second differences*

$$\Delta^2 p_n := \Delta p_n - \Delta p_{n-1} = p_n - 2p_{n-1} + p_{n-2}$$

are nonnegative. In this case, one can conveniently recompile the Smith entries of $DU_n + tI$ over $\mathbb{Z}[t]$ from (2.1) and express their multiplicities in terms of these second differences:

entry	multiplicity
$\begin{cases} (n+1)!_{r,t} & \text{if } r \geq 2 \\ (n+1+t) \cdot (n-1)!_{1,t} & \text{if } r = 1 \end{cases}$	1
$(i+1)!_{r,t}$	$\Delta^2 p_{n-i}$ for $0 \leq i \leq n-1$
1	p_{n-1}

where

$$\ell_{r,t} := (r \cdot \ell + t)(r \cdot (\ell - 1) + t) \cdots (r \cdot 1 + t)$$

and where by convention we set $p_k = 0$ if k is negative.

2.2. On the up and down maps themselves.

Conjecture 2.4. *For every n , the matrix U_n in $\mathbb{Z}^{p_{n+1} \times p_n}$ has only ones and zeroes in its Smith form over \mathbb{Z} , that is, $\text{coker}(U_n : \mathbb{Z}^{p_n} \rightarrow \mathbb{Z}^{p_{n+1}})$ is free abelian.*

Note that it would be equivalent to conjecture that D_{n+1} has free cokernel, or only ones and zeroes in its Smith form of \mathbb{Z} , since $D_{n+1} = U_n^t$.

Proposition 2.5. *Whenever Conjecture 2.3 holds, then Conjecture 1.1 implies Conjecture 2.4.*

Proof. One can be more precise about the relationship: whenever Conjecture 2.3 holds, one has equivalence between the assertion of Conjecture 2.4 and the correctness of the Smith form over \mathbb{Z} for UD_n which is predicted by setting $t = 0$ in the Smith form over $\mathbb{Z}[t]$ for $UD_n + tI$ that follows from Conjecture 1.1 and Proposition 8.4.

To see this, recall that Proposition 2.1 allows one to rephrase Conjecture 2.3 as saying that the zero eigenvalue for UD_n achieves the largest multiplicity among all of its non-unit eigenvalues. Thus whenever this holds, Proposition 8.4 says that the Smith form over \mathbb{Z} for UD_n predicted by Conjecture 1.1 will have only zeroes and ones as the diagonal entries. This is equivalent to $\text{coker}(UD_n) = \mathbb{Z}^{p_n - p_{n-1}}$, which by Proposition 8.3 is equivalent to $\text{coker}(U) = \mathbb{Z}^{p_n - p_{n-1}}$, that is, to Conjecture 2.4. \square

Remark 2.6. Note that Conjecture 2.4 is equivalent (via Proposition 8.1) to saying that the gcd of all of the maximal minor subdeterminants in U_n or D_{n+1} is 1. In fact, many examples exhibit a strictly stronger property: U_n has at least one maximal minor subdeterminant *equal* to ± 1 . We suspect that this stronger property does *not* hold for all r -differential posets, although we have not yet encountered any counterexamples

3. COMPUTER TESTS

We report here on a few numerical experiments.

The first author and P. Byrnes used a computer to find all possible *partial 1-differential posets* (in the sense described in Section 4.1 below)

- of rank 8, coming in 643 poset isomorphism classes, and
- of rank 9, coming in 44605 poset isomorphism classes.

Any of these partial 1-differential posets can be extended to a full 1-differential poset by Wagner's construction from the next section, and therefore represent all the possibilities for the first 8 or 9 ranks.

The data up through rank 9 exhibits three isomorphism classes where the first differences of the rank numbers are *not* weakly increasing; see the remark at the end of Section 2.1 for the significance. All three of these isomorphism classes have rank sizes

$$(p_0, p_1, p_2, \dots, p_9) = (1, 1, 2, 3, 5, 7, 11, 15, 26, 36)$$

whose first differences are

$$(\Delta p_1, \Delta p_2, \dots, \Delta p_9) = (0, 1, 1, 2, 2, 4, 4, 11, 10)$$

so that $\Delta_8 = 11 > 10 = \Delta_9$.

Regarding Conjecture 1.1, if it were correct, then the Smith form over $\mathbb{Z}[t]$ would have its diagonal Smith entries $s_i(t)$ given explicitly by (2.1), via Proposition 2.1 and Proposition 8.4. As mentioned in Remark 8.7 of the Appendix, this then predicts the Smith form over \mathbb{Z} (which always exists) for $DU_n + kI$ when k is any chosen integer. For each of the 643 non-isomorphic partial 1-differential posets of rank 8, considering $DU_n + kI$ with $n = 0, 1, 2, \dots, 7$, this predicted Smith form over \mathbb{Z} was checked to be correct for all integers k in the range $[-10, +10]$. In addition, in Young's lattice \mathbf{Y} , the Smith form for $DU_n (= DU_n + 0 \cdot I)$ was checked to obey the prediction up through $n = 20$.

4. CONSTRUCTIONS FOR r -DIFFERENTIAL POSETS

We recall here two known ways to construct new differential posets from old ones, and discuss how these constructions respect Conjectures 1.1, 2.3, and 2.4.

4.1. Wagner's reflection-extension construction. *Wagner's construction* produces an r -differential posets from a *partial r -differential poset* of some finite rank (see [19, §6]). Let P be a finite graded poset of rank n with a minimum element, and assume that for $i = 0, 1, \dots, n-1$ one has

$$DU_i - UD_i = rI.$$

One calls P a *partial r -differential poset* of rank n . Let P^+ be the poset of rank $n+1$ obtained from P in the following way: for each $v \in P_{n-1}$, add a vertex v^* of rank $n+1$ to P that covers exactly those $x \in P_n$ covering v . Finally, above each

$x \in P_n$ we adjoin r new elements covering only x in P^+ . It is not hard to see that iterating this construction produces an r -differential poset.

An important example of this construction are the posets $Z(r)$ described explicitly in Section 5 below, but which can also be obtained by applying Wagner's construction to the partial poset of rank zero having only one element. As an example, see Figure 1 for $Z(2)$.

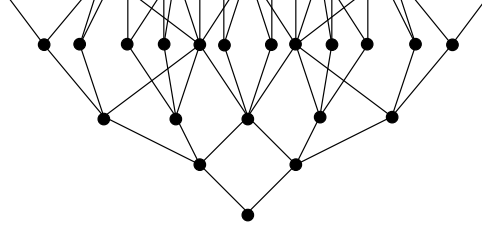


FIGURE 1. Part of the 2-differential poset $Z(2)$.

Proposition 4.1. *Let \hat{P} be a partial r -differential poset of rank n , and P the differential poset obtained from \hat{P} by Wagner's construction.*

- (i) *If the inequalities of Conjecture 2.3 are satisfied by \hat{P} up through rank n , then P satisfies Conjecture 2.3.*
- (ii) *If Conjecture 2.4 is satisfied by \hat{P} , in the sense that $\text{coker}(U_j)$ is free for all $0 \leq j \leq n-1$, then P satisfies Conjecture 2.4.*

Proof. For (i), note that

$$p_{n+1} = p_{n-1} + rp_n$$

and hence

$$\Delta p_{n+1} = (r-1)p_n + p_{n-1}.$$

Therefore if $r \geq 2$ then

$$\Delta p_{n+1} \geq p_n \geq \Delta p_n$$

giving Conjecture 2.3 in this case. If $r = 1$ then it says

$$\Delta p_{n+1} = p_{n-1} \geq p_{n-j} \geq \Delta p_{n-j}$$

for $j \geq 1$, which gives Conjecture 2.3 in this case.

For (ii), note that by Wagner's construction, the $p_{n+1} \times p_n$ matrix for U_n will contain a $p_n \times p_n$ (maximal) minor equal to the identity matrix, and hence have free cokernel. \square

Remark 4.2. The proof of Theorem 5.1 below will show that, for $r \geq 2$, Wagner's construction also respects Conjecture 1.1, in the following sense: if one has a partial r -differential poset with ranks $0, 1, \dots, n-1$, and one applies Wagner's construction twice producing the two new ranks P_n, P_{n+1} , then one has the following relation (see (5.3) below):

$$DU_n + tI \sim I_{n-1} \oplus (t+r)I_{(r-2) \cdot p_{n-1} + p_{n-2}} \oplus (t+r)(DU_{n-1} + (t+r)I)$$

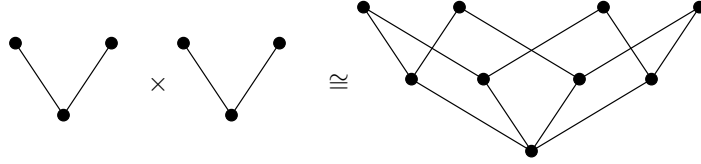
where $A \oplus B$ denotes block direct sum of matrices, and $A \sim B$ for matrices A, B in $\mathbb{Z}[t]^{m \times n}$ means that $P(t) \cdot A \cdot Q(t) = B$ for some $P(t), Q(t)$ in $GL_m(\mathbb{Z}[t]), GL_n(\mathbb{Z}[t])$

respectively. Thus $DU_n + tI$ has Smith form over $\mathbb{Z}[t]$ if and only if $DU_{n-1} + tI$ does.

Unfortunately the part of the proof of Theorem 5.1 which treats the case $r = 1$ does not have this nice property.

4.2. The Cartesian product construction. If P and P' are posets, their *Cartesian product* $P \times P'$ is endowed with the componentwise partial order.

Example 4.3. The following is a small example for the Cartesian product of two posets.



Stanley observed that if P and P' are r - and r' -differential posets, then $P \times P'$ is an $(r + r')$ -differential poset.

Proposition 4.4. *If P, P' are any r -, r' -differential posets, then Conjecture 2.3 holds for the $(r + r')$ -differential poset $P \times P'$.*

Proof. As $r + r' \geq 2$, one needs to show that the rank sizes $r''_n := |(P \times P')_n|$ satisfy $\Delta^2 r''_n \geq 0$. If

$$\begin{aligned} P(t) &:= \sum_n r_n t^n \\ P'(t) &:= \sum_n r'_n t^n \\ \text{then } (P \times P')(t) &:= \sum_n r''_n t^n = P(t)P'(t). \end{aligned}$$

Since both P, P' have their rank sizes weakly increasing by Proposition 2.1, both $(1 - t)P(t)$ and $(1 - t)P'(t)$ lie in $\mathbb{N}[[t]]$. Hence their product

$$(1 - t)^2 P(t)P'(t) = (1 - 2t + t^2) \sum_n r''_n t^n$$

also lies in $\mathbb{N}[[t]]$, which means that $\Delta^2 r''_n \geq 0$. □

Proposition 4.5. *Let P and P' be differential posets. If Conjecture 2.4 holds for either P or P' then it also holds for $P \times P'$.*

Proof. We begin by noting that one can identify

$$\mathbb{Z}^{(P \times P')_n} = \bigoplus_{i+j=n} \mathbb{Z}^{P_i} \otimes_{\mathbb{Z}} \mathbb{Z}^{P'_j}.$$

Under this identification, the up map $U_{n-1}^{P \times P'} : \mathbb{Z}^{(P \times P')_{n-1}} \rightarrow \mathbb{Z}^{(P \times P')_n}$ is given by

$$(4.1) \quad \bigoplus_{i+j=n} (I_{P_i} \otimes U'_j) \oplus (U_i \otimes I_{P'_j})$$

or in block matrix terms:

$$U_{n-1}^{P \times P'} = \begin{bmatrix} I_{p_0} \otimes U'_{n-1} & & & & \\ U_0 \otimes I_{p'_{n-1}} & I_{p_1} \otimes U'_{n-2} & & & \\ & U_1 \otimes I_{p'_{n-2}} & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & I_{p_{n-1}} \otimes U'_0 \\ & & & & U_{n-1} \otimes I_{p'_0} \end{bmatrix}.$$

Assume Conjecture 2.4 holds for P (respectively, for P'). Then one can perform row and column operations to arrive at a matrix with the same zero blocks, but where the (i, i) block of the form $I_{p_i} \otimes U'_{n-i-1}$ (respectively, the $(i+1, i)$ block of the form $U_i \otimes I_{p'_{n-i-1}}$) has been changed to

$$(4.2) \quad I_{p_i} \otimes \begin{bmatrix} I_{p'_{n-i-1}} \\ 0 \end{bmatrix}. \quad \left(\text{respectively, } \begin{bmatrix} I_{p_i} & 0 \end{bmatrix} \otimes I_{p'_{n-i-1}} \right).$$

One can then see that no matter how the factor below it (respectively, to the right of it) has been altered during this process, one can then use the new block in (4.2) to eliminate it with row (respectively, column) operations. Thus the whole matrix has only zeroes and ones in its Smith form over \mathbb{Z} , and hence has free cokernel, as desired. \square

Remark 4.6. The expression (4.1) for the up map in $P \times P'$ can also be used to give the following block tridiagonal form for DU_n in the Cartesian product $P \times P'$:

$$DU_n^{P \times P'} = D_n \otimes U'_n \oplus (I_{p_n} \otimes DU'_n + DU_n \otimes I_{p'_n}) \oplus U_n \otimes D'_n.$$

However, we have not seen how to use this to show that the Cartesian product construction respects Conjecture 1.1 in some way.

5. THE YOUNG-FIBONACCI POSETS

As a set, the *Fibonacci r -differential poset* $Z(r)$ consists of all finite words using alphabet $\{1_1, 1_2, \dots, 1_r, 2\}$. For two words $w, w' \in Z(r)$, we define w to cover w' if either

- (1) w' is obtained from w by changing a 2 to some 1_i , as long as only 2's occur to its left, or
- (2) w' is obtained from w by deleting its first letter of the form 1_i .

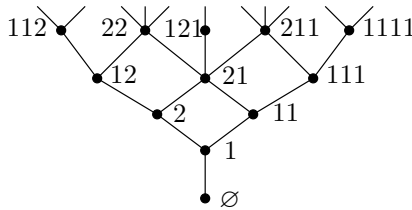


FIGURE 2. The Young-Fibonacci lattice \mathbf{YF} .

It is known and not hard to show [19, §6] that $Z(r)$ is obtained using Wagner's construction from a single point and hence an r -differential poset; see The reason

for its name is that the rank sizes of $Z(1) = \mathbf{YF}$ (the *Young-Fibonacci* lattice) are the Fibonacci numbers; see Figure 2.

5.1. Proof of all conjectures for \mathbf{YF} and $Z(r)$.

Theorem 5.1. *All of the Conjectures 1.1, 2.3, 2.4 hold for the Young-Fibonacci r -differential posets $Z(r)$ with $r \geq 1$.*

Proof. It follows from Proposition 4.1 that Conjectures 2.3 and 2.4 hold. For Conjecture 1.1, one begins with the following recursive descriptions of U_n and D_n :

$$(5.1) \quad U_n = \begin{bmatrix} D_n \\ I_{p_n} \\ \vdots \\ I_{p_n} \end{bmatrix} \quad \text{and} \quad D_{n+1} = \begin{bmatrix} U_{n-1} & I_{p_n} & \cdots & I_{p_n} \end{bmatrix}.$$

Note that each block matrix above contains exactly r blocks of I_{p_n} . Using this, one can write

$$D_{n+1} = \begin{bmatrix} D_{n-1} & I_{p_{n-2}} & & & I_{p_{n-2}} & & \\ I_{p_{n-1}} & & I_{p_{n-1}} & & & I_{p_{n-1}} & \\ \vdots & & & \ddots & & & \ddots \\ I_{p_{n-1}} & & & & I_{p_{n-1}} & & I_{p_{n-1}} \end{bmatrix}$$

and multiplying it by its transpose U_n , one obtains the following $(r+1) \times (r+1)$ block matrix for $n \geq 2$:

$$(5.2) \quad \begin{bmatrix} DU_{n-2} + tI_{p_{n-2}} & D_{n-1} & \cdots & \cdots & \cdots & D_{n-1} \\ U_{n-2} & (t+1)I_{p_{n-1}} & I_{p_{n-1}} & \cdots & \cdots & I_{p_{n-1}} \\ \vdots & I_{p_{n-1}} & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \ddots & I_{p_{n-1}} \\ U_{n-2} & I_{p_{n-1}} & \cdots & \cdots & I_{p_{n-1}} & (t+1)I_{p_{n-1}} \end{bmatrix}.$$

We now use this description of $DU_n + (t-r)I_{p_n}$ together with elementary row and column operations to show that it has a Smith form in $\mathbb{Z}[t]$. Given two matrices A, B in $\mathbb{Z}[t]^{m \times n}$, say that $A \sim B$ if there exist $P(t), Q(t)$ in $GL_m(\mathbb{Z}[t]), GL_n(\mathbb{Z}[t])$ with $PAQ = B$.

Case 1. $r \geq 2$

Because $r \geq 2$, in (5.2) there is a copy of $I_{p_{n-1}}$ in the second row and last column. Use it to first clear out the blocks in the second row, then to clear out the blocks

in the last columns, and lastly, negate the second column:

$$\begin{aligned}
& DU_n + (t-r)I_{p_n} \\
& \sim \begin{bmatrix} tI_{p_{n-2}} & -tD_{n-1} & 0 & \cdots & \cdots & 0 & D_{n-1} \\ 0 & 0 & 0 & & & \vdots & I_{p_{n-1}} \\ 0 & -tI_{p_{n-1}} & tI_{p_{n-1}} & \ddots & & \vdots & I_{p_{n-1}} \\ 0 & -tI_{p_{n-1}} & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & tI_{p_{n-1}} & 0 & I_{p_{n-1}} \\ 0 & -tI_{p_{n-1}} & 0 & \cdots & 0 & tI_{p_{n-1}} & I_{p_{n-1}} \\ -tU_{n-2} & -t(t+2)I_{p_{n-1}} & -tI_{p_{n-1}} & \cdots & -tI_{p_{n-1}} & -tI_{p_{n-1}} & (t+1)I_{p_{n-1}} \end{bmatrix} \\
& \sim \begin{bmatrix} tI_{p_{n-2}} & tD_{n-1} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & & & \vdots & I_{p_{n-1}} \\ 0 & tI_{p_{n-1}} & tI_{p_{n-1}} & \ddots & & \vdots & 0 \\ 0 & tI_{p_{n-1}} & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & tI_{p_{n-1}} & 0 & 0 \\ 0 & tI_{p_{n-1}} & 0 & \cdots & 0 & tI_{p_{n-1}} & 0 \\ -tU_{n-2} & t(t+2)I_{p_{n-1}} & -tI_{p_{n-1}} & \cdots & -tI_{p_{n-1}} & -tI_{p_{n-1}} & 0 \end{bmatrix}
\end{aligned}$$

After rearranging the rows and columns, this yields

$$I_{p_{n-1}} \oplus \begin{bmatrix} tD_{n-1} & 0 & 0 & \cdots & 0 & tI_{p_{n-2}} \\ tI_{p_{n-1}} & tI_{p_{n-1}} & 0 & \cdots & 0 & 0 \\ tI_{p_{n-1}} & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & 0 \\ tI_{p_{n-1}} & 0 & \cdots & 0 & tI_{p_{n-1}} & 0 \\ t(t+2)I_{p_{n-1}} & -tI_{p_{n-1}} & \cdots & -tI_{p_{n-1}} & -tI_{p_{n-1}} & -tU_{n-2} \end{bmatrix}$$

Subtracting columns 2, 3, ..., $r-1$ each from the first column yields

$$\begin{aligned}
& I_{p_{n-1}} \oplus \begin{bmatrix} tD_{n-1} & 0 & \cdots & \cdots & 0 & tI_{p_{n-2}} \\ 0 & tI_{p_{n-1}} & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & tI_{p_{n-1}} & 0 \\ t(r+t)I_{p_{n-1}} & 0 & \cdots & \cdots & 0 & -tU_{n-2} \end{bmatrix} \\
& \sim I_{p_{n-1}} \oplus tI_{(r-2) \cdot p_{n-1}} \oplus \begin{bmatrix} tD_{n-1} & tI_{p_{n-2}} \\ t(r+t)I_{p_{n-1}} & -tU_{n-2} \end{bmatrix} \\
& \sim I_{p_{n-1}} \oplus tI_{(r-2) \cdot p_{n-1}} \oplus tI_{p_{n-2}} \oplus t((r+t)I_{p_{n-1}} + UD_{n-1}) \\
& = I_{p_{n-1}} \oplus tI_{(r-2) \cdot p_{n-1} + p_{n-2}} \oplus t(DU_{n-1} + tI_{p_{n-1}}),
\end{aligned}$$

for $n \geq 2$.

That is, when $r \geq 2$, one has for $n \geq 2$ that

$$(5.3) \quad DU_n + tI \sim I_{p_{n-1}} \oplus (t+r)I_{(r-2) \cdot p_{n-1} + p_{n-2}} \oplus (t+r)(DU_{n-1} + (t+r)I).$$

This shows that $DU_n + tI$ has a Smith form over $\mathbb{Z}[t]$ by induction on n .

Case 2. $r = 1$

For $r = 1$, one proceeds slightly differently. One can check that $DU_n + tI$ has a Smith form over $\mathbb{Z}[t]$ in the base cases of $n = 0, 1, 2, 3$ directly. Then for $n \geq 4$, starting with (5.2) for $r = 1$, one has

$$\begin{aligned} & DU_n + (t-1)I_{p_n} \\ &= \begin{bmatrix} DU_{n-2} + tI_{p_{n-2}} & D_{n-1} \\ U_{n-2} & (t+1)I_{p_{n-1}} \end{bmatrix} \\ &= \begin{bmatrix} DU_{n-2} + tI_{p_{n-2}} & U_{n-3} & I_{p_{n-2}} \\ D_{n-2} & (t+1)I_{p_{n-3}} & 0 \\ I_{p_{n-2}} & 0 & (t+1)I_{p_{n-2}} \end{bmatrix} \end{aligned}$$

in which the second step used the recursive descriptions (5.1) of U_n, D_n . Use the $I_{p_{n-2}}$ in the upper-right block to clear out the first row, and then the first column:

$$\begin{aligned} & DU_n + (t-1)I_{p_n} \\ &\sim \begin{bmatrix} 0 & 0 & I_{p_{n-2}} \\ D_{n-2} & (t+1)I_{p_{n-3}} & 0 \\ -(t+1)DU_{n-2} - (t^2 + t - 1)I_{p_{n-2}} & -(t+1)U_{n-3} & (t+1)I_{p_{n-2}} \end{bmatrix} \\ &\sim \begin{bmatrix} 0 & 0 & I_{p_{n-2}} \\ D_{n-2} & (t+1)I_{p_{n-3}} & 0 \\ -(t+1)DU_{n-2} - (t^2 + t - 1)I_{p_{n-2}} & -(t+1)U_{n-3} & 0 \end{bmatrix} \\ &\sim I_{p_{n-2}} \oplus A \end{aligned}$$

where

$$A := \begin{bmatrix} D_{n-2} & (t+1)I_{p_{n-3}} \\ -(t+1)DU_{n-2} - (t^2 + t - 1)I_{p_{n-2}} & -(t+1)U_{n-3} \end{bmatrix}$$

Next perform the column operation which subtracts from the first column the result of right-multiplying the second column by D_{n-2} . Using the fact that $UD_{n-2} = DU_{n-2} - I_{p_{n-2}}$, this yields a matrix

$$A_n(t) := \begin{bmatrix} -tD_{n-2} & (t+1)I_{p_{n-3}} \\ -t(t+2)I_{p_{n-2}} & -(t+1)U_{n-3} \end{bmatrix}$$

It only remains to show that $A_n(t)$ has a Smith form over $\mathbb{Z}[t]$, which we do by induction on n . Start by using the recursive descriptions (5.1) of D_n, U_n to rewrite $A_n(t)$ as

$$\begin{aligned} A_n(t) &:= \begin{bmatrix} -tU_{n-4} & -tI_{p_{n-3}} & (t+1)I_{p_{n-3}} \\ -t(t+2)I_{p_{n-4}} & 0 & -(t+1)D_{n-3} \\ 0 & -t(t+2)I_{p_{n-4}} & -(t+1)I_{p_{n-3}} \end{bmatrix} \\ &\sim \begin{bmatrix} -tU_{n-4} & -tI_{p_{n-3}} & I_{p_{n-3}} \\ -t(t+2)I_{p_{n-4}} & 0 & -(t+1)D_{n-3} \\ 0 & -t(t+2)I_{p_{n-4}} & -(t^2 + 3t + 1)I_{p_{n-3}} \end{bmatrix} \end{aligned}$$

where the latter equivalence comes from adding the second column to the third. Now one can use the $I_{p_{n-3}}$ in the upper right block to clear out the first row, and then clear out the last column. The end result shows that

$$A_n(t) \sim I_{n_3} \oplus (-t) \begin{bmatrix} (t+2)I_{p_{n-4}} + (t+1)DU_{n-4} & (t+1)D_{n-3} \\ (t^2 + 3t + 1)U_{n-4} & (t^2 + 4t + 3)U_{n-3} \end{bmatrix}.$$

If one now subtracts from the first column the result of right-multiplying the second column by U_{n-4} , one concludes that

$$\begin{aligned} A_n(t) &\sim I_{n_3} \oplus (-t) \begin{bmatrix} (t+2)I_{p_{n-4}} & (t+1)D_{n-3} \\ -(t+2)U_{n-4} & (t+1)(t+3)U_{n-3} \end{bmatrix} \\ &\sim I_{n_3} \oplus (-t)A_{n-1}(t+1). \end{aligned}$$

Since $A_{n-1}(t)$ (and hence also $A_{n-1}(t+1)$) has a Smith form over $\mathbb{Z}[t]$ by induction, so does $A_n(t)$. \square

6. YOUNG'S LATTICE AND ITS CARTESIAN PRODUCTS

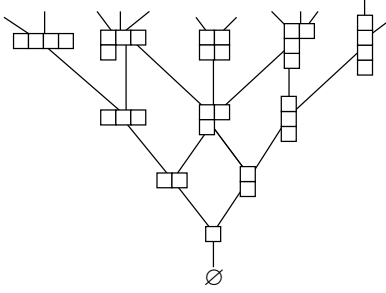


FIGURE 3. Young's lattice.

The prototypical example of a 1-differential poset is *Young's lattice*, denoted by \mathbf{Y} . This is the set of all number partitions, ordered by inclusion of Young diagrams; see Figure 3. Its r -fold Cartesian product \mathbf{Y}^r is then an r -differential poset. These give the primary example of a set-up which we define next.

Definition 6.1. For r a positive integer, define an r -differential tower of groups to be an infinite tower of finite groups $1 = G_0 \leq G_1 \leq G_2 \leq \cdots$ such that the branching rules for restricting irreducibles from G_n to G_{n-1} are *multiplicity-free*, and satisfy

$$(6.1) \quad \text{Res}_{G_n}^{G_{n+1}} \text{Ind}_{G_n}^{G_{n+1}} - \text{Ind}_{G_{n-1}}^{G_n} \text{Res}_{G_{n-1}}^{G_n} = r \cdot \text{id}.$$

Here the maps in question are considered as operators on the \mathbb{Z} -module of virtual characters $\mathbf{Cl}_{\mathbb{Z}}[G_n]$, that is, the \mathbb{C} -valued class functions on G_n that are \mathbb{Z} -linear combinations of irreducible characters for G_n .

Each r -differential tower of groups gives rise to an r -differential poset P , whose n^{th} rank P_n is labelled by $\text{Irr}(G_n)$, the set of irreducible representations of G_n . One includes a covering relation between a G_{n-1} -irreducible χ and a G_n -irreducible ψ exactly when $\text{Res}_{G_{n-1}}^{G_n} \psi$ contains χ (with multiplicity one) as an irreducible constituent. Identifying \mathbb{Z}^{P_n} with $\mathbf{Cl}_{\mathbb{Z}}[G_n]$, equation (6.1) becomes $DU_n - UD_n = rI$.

Example 6.2. Let G be a finite abelian group of order r , and let G_n be the wreath product $G \wr \mathfrak{S}_n = \mathfrak{S}_n[G]$ of G and the symmetric group \mathfrak{S}_n . An irreducible representation of G_n can naturally be indexed by an element of \mathbf{Y}^r at rank n , that is, by an r -tuple $(\lambda^{(1)}, \dots, \lambda^{(r)})$ of partitions whose total weight $\sum_i |\lambda^{(i)}|$ equals n . Furthermore, the branching rule from G_n to G_{n-1} is multiplicity free: one obtains a sum of irreducibles indexed by all r -tuples obtained by removing one

box from any of the $\lambda^{(i)}$; see Macdonald [11, Chapter 1, Appendix B], Okada [17, Corollary to Theorem 4.1], and Stembridge [23, Lemma 7.1]. Thus the irreducibles and branching in this tower of groups are described by the r -differential poset \mathbf{Y}^r , and hence this is an r -differential tower of groups.

Remark 6.3. A recent result of Bergeron, Lam, and Li [1] implies that every r -differential tower of groups that satisfies some mild extra hypotheses will have $|G_n| = r^n n!$. This means that these structures are more rigid than one might expect.

For the purposes of some of the proofs in this section, we briefly review in the case $r = 1$, so that $G_n = \mathfrak{S}_n$, some facts about bases of $\mathbf{Cl}_{\mathbb{Z}}[\mathfrak{S}_n]$ and their labelling by symmetric functions as in [11] and [22, Chapter 7]; see these references for more details, as well as [19, Remark on p. 926].

The irreducible (complex) characters of \mathfrak{S}_n are indexed by partitions λ of n , and are associated via Frobenius' characteristic map to the Schur functions s_λ . The above discussion says that the branching map D_n sends the basis element indexed by s_λ to the sum of all s_μ where μ is obtained from λ by removing one cell from the Ferrers diagram. On the other hand, there is another \mathbb{Z} -basis for $\mathbf{Cl}_{\mathbb{Z}}[\mathfrak{S}_n]$ indexed by partitions λ of n , given by the characters of the coset representations for Young subgroups $\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_\ell}$ inside \mathfrak{S}_n . The characters of these coset representations correspond under the Frobenius characteristic map with the symmetric functions $h_\lambda := h_{\lambda_1} \cdots h_{\lambda_\ell}$ where h_r is the complete homogeneous symmetric function of degree r . In this basis, since the induction map U_n multiplies by $s_1 = h_1$, it sends h_λ to $h_{(\lambda,1)}$. Also, the branching rule D_n sends h_n to h_{n-1} and then extends as a derivation on products. From this one can derive the following special case of the Mackey formula: if $\lambda = (\lambda_1, \dots, \lambda_\ell) \vdash n$, then

$$(6.2) \quad DU(h_\lambda) = D(h_1 h_\lambda) = h_\lambda + h_1 \sum_{i=1}^{\ell} h_{(\lambda_1, \dots, \lambda_i-1, \dots, \lambda_\ell)} = h_\lambda + \sum_{i=1}^{\ell} h_{(\lambda_1, \dots, \lambda_i-1, \dots, \lambda_\ell, 1)}.$$

6.1. Proof of Conjectures 2.3 and 2.4 for \mathbf{Y}^r .

Proposition 6.4. *Conjecture 2.3 holds for \mathbf{Y}^r*

Proof. By Proposition 4.4, it is enough to check it for $r = 1$, that is, for \mathbf{Y} . We will show the stronger assertion that

$$(6.3) \quad \Delta p_1 \leq \Delta p_2 \leq \cdots.$$

Recall that p_n counts the number of partitions of n . One can inject the partitions of $n-1$ into the partitions of n by adding a part of size 1. Hence $\Delta p_n = p_n - p_{n-1}$ counts the number of partitions of n that contain no parts of size 1.

One can then inject such partitions with no parts of size 1 for $n-1$ into those for n , by adding 1 to the largest part. This shows $\Delta p_{n-1} \leq \Delta p_n$. \square

Proposition 6.5. *Conjecture 2.4 holds for \mathbf{Y}^r*

Proof. By Proposition 4.5, it is enough to check it for \mathbf{Y} . But when one uses the \mathbb{Z} -basis for $\mathbf{Cl}_{\mathbb{Z}}(\mathfrak{S}_n)$ indexed by $\{h_\lambda\}$ as discussed above, then the fact that $U_n(h_\lambda) = h_{(\lambda,1)}$ shows that U_n has free cokernel. \square

6.2. Ones in the Smith form of $DU + kI$ in \mathbf{Y} . Conjecture 1.1 predicts for each integer k the Smith normal form over \mathbb{Z} for $DU + kI$; see Proposition 8.4. In particular, it predicts the number of *ones* that should appear in these Smith normal forms. We check here that those predictions are correct for Young's lattice \mathbf{Y} .

Proposition 6.6. *In Young's lattice \mathbf{Y} , for every integer k , the number of ones in the Smith normal form over \mathbb{Z} of $DU_n + kI$ is*

$$(6.4) \quad \begin{cases} p_{n-1} & \text{for } k \neq 0, -2 \\ p_n - p_{n-1} + p_{n-2} & \text{for } k = 0, -2, \end{cases}$$

which agrees with the prediction from Conjecture 1.1.

Proof. Let $\text{ones}(A)$ for an integer matrix A denote the number of ones in its Smith normal form over \mathbb{Z} .

First we check that the numbers given in (6.4) are the same as the ones predicted by the conjecture. Proposition 8.4 predicts that $\text{ones}(DU_n + kI)$ should be the difference between p_n and the highest multiplicity attained by a non-unit (that is, not ± 1) eigenvalue of $DU_n + kI$. Because \mathbf{Y} satisfies the inequalities (6.3), Proposition 2.1 shows that the highest multiplicity of a non-unit eigenvalue will either be the multiplicity

- $p_n - p_{n-1}$ attained by $k + 1$ whenever $k + 1 \neq \pm 1$, or
- $p_{n-1} - p_{n-2}$ attained by $k + 2$.

Thus the predictions agree with (6.4).

To show that the numbers in (6.4) actually match $\text{ones}(DU_n + kI)$, we express $DU_n + kI$ with respect to two different (ordered) bases of $\mathbf{Cl}_{\mathbb{Z}}(\mathfrak{S}_n)$.

First use the $\{s_\lambda\}$ or irreducible character basis, with the partitions ordered lexicographically. One can see that it contains a $p_{n-1} \times p_{n-1}$ lower unitriangular submatrix whose rows (resp. columns) are indexed by the partitions $(\mu_1, \dots, \mu_\ell, 1)$ (resp. $(\mu_1 + 1, \mu_2, \mu_3, \dots, \mu_\ell)$) where μ runs through all partitions of $n - 1$. For example, when $n = 5$, the matrix looks as follows, with the rows and columns of the aforementioned submatrix indicated by asterisks:

$$\begin{array}{c} \begin{matrix} & 5 & 41^* & 32 & 311^* & 221^* & 2111^* & 11111^* \end{matrix} \\ \begin{matrix} 5^* \\ 41^* \\ 32^* \\ 311^* \\ 221 \\ 2111^* \\ 11111 \end{matrix} \begin{pmatrix} k+2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & k+3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & k+3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & k+3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & k+3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & k+3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & k+2 \end{pmatrix} \end{array}$$

Note that this shows, via Proposition 8.1, that for any integer k one has the inequality $\text{ones}(DU_n + kI) \geq p_{n-1}$.

Alternatively, one can express $DU_n + kI$ in the $\{h_\lambda\}$ -basis, ordered so that the partitions λ having no part of size 1 come first. Then equation 6.2 shows that it will have this block upper-triangular form:

$$(6.5) \quad \begin{bmatrix} (k+1)I_{p_n - p_{n-1}} & 0 \\ * & [DU_{n-1}] + (k+1)I_{p_{n-1}} \end{bmatrix}.$$

For $n = 5$, this looks as follows:

$$\begin{array}{c} 5 \quad 32 \quad 41 \quad 2^2 1 \quad 31^2 \quad 21^3 \quad 1^5 \\ \begin{pmatrix} k+1 & & & & & & \\ 0 & k+1 & & & & & \\ 1 & 0 & k+2 & & & & \\ 0 & 1 & 0 & k+2 & & & \\ 0 & 1 & 1 & 0 & k+3 & & \\ 0 & 0 & 0 & 2 & 1 & k+4 & \\ 0 & 0 & 0 & 0 & 0 & 1 & k+6 \end{pmatrix}, \end{array}$$

Since the first $p_n - p_{n-1}$ rows of the matrix for $DU_n + kI$ in (6.5) have every entry divisible by $k+1$, any minor subdeterminant of size $p_{n-1} + 1$ will be divisible by $k+1$. Consequently, by Proposition 8.1, the Smith entry $s_{p_{n-1}+1} \neq 1$ as long as $k+1$ is not a unit of \mathbb{Z} , that is, as long as $k \neq 0, -2$. This shows $\text{ones}(DU_n + kI) \leq p_{n-1}$ in this case, and hence $\text{ones}(DU_n + kI) = p_{n-1}$ in light of the above lower bound.

Now if $k = 0$ or -2 then (6.5) shows

$$\text{ones}(DU + kI) = p_n - p_{n-1} + \text{ones}(DU_{n-1} + (k+1)I).$$

Therefore it only remains to check that for $k = 0$ and for $k = -2$ one has $\text{ones}(DU_{n-1} + (k+1)I) = p_{n-2}$:

- For $k = 0$ this was just proven above, replacing n by $n-1$ and taking $k = 1$.
- For $k = -2$ one has

$$\text{ones}(DU_{n-1} - I) = \text{ones}(UD_{n-1}) = p_{n-2}$$

where the second equality holds because Propositions 6.5 and 8.3 tell us that UD_{n-1} has only zeroes and ones in its Smith form.

□

Remark 6.7. It seems likely that one can extend the statement of Proposition 6.6 and its proof from \mathbf{Y} to \mathbf{Y}^r , using the analogues of the $\{h_\lambda\}$ basis for $\mathbf{CI}_{\mathbb{Z}}(G_n)$ where $G_n = \mathbb{Z}/r\mathbb{Z} \wr \mathfrak{S}_n$, as discussed in [11, Chapter 1, Appendix B, §8]. However, we have not seriously attempted to carry this out.

One slight annoyance is that the general prediction implied by Conjecture 1.1 for the number of ones when $r \geq 2$ will have a further special case in its statement: when $r = 2$ and $k = -3$, the two eigenvalues $r+k = -1$ and $2r+k = +1$ achieving the highest multiplicities *both* turn out to be units in \mathbb{Z}^\times . Thus Conjecture 1.1 predicts, via Proposition 8.4, that

$$\text{ones}(DU_n + kI) \begin{cases} p_{n-1} & \text{for } r+k \neq \pm 1 \\ p_n - p_{n-1} + p_{n-2} & \text{for } r+k = \pm 1 \text{ and } (r,k) \neq (2,-3) \\ p_n - p_{n-2} + p_{n-3} & \text{for } (r,k) = (2,-3). \end{cases}$$

Remark 6.8. Is there hope to extend the previous results to r -differential towers of group algebras, or to r -differential self-dual Hopf algebras, as discussed in [1]?

In fact, Conjecture 2.4 is essentially asking for something like an $\{h_\lambda\}$ basis in these contexts. U_n having free cokernel is equivalent to saying that there exists a choice of bases in the source and target so that that map $U_n : \mathbb{Z}^{P_n} \rightarrow \mathbb{Z}^{P_{n+1}}$ sends basis elements into basis elements, as in $U_n(h_\lambda) = h_{(\lambda,1)}$.

6.3. Last Smith entry of $DU + kI$ in \mathbf{Y}^r and towers of group algebras.

We wish to show that for \mathbf{Y}^r , and more generally, for any r -differential poset coming from a tower of finite groups, the last Smith invariants s_{p_n} for either of the operators $DU_n + kI$ or $UD_n + kI$ match the prediction provided by Conjecture 1.1 via Proposition 8.4.

The idea is to use Corollary 8.2. We start with some facts about the spectrum and inverse of UD when $D := \text{Res}_H^G$ and $U = \text{Ind}_H^G$ are restriction and induction of class functions for *any* pair of nested finite groups $H \subseteq G$. Part (i) of the next proposition is a generalization of Stanley's [19, Proposition 4.7] that was first observed by Okada [17, Appendix, Theorem A].

Proposition 6.9. *Let G be a finite group, and $H \subseteq G$ a subgroup.*

- (i) *The operator UD on $\mathbf{Cl}_{\mathbb{Z}}(G)$ has a complete system of orthogonal eigenvectors given by*

$$v^{(g)} := \sum_{\chi \in \text{Irr}(G)} \chi(g) \cdot \chi$$

as g ranges through a set of representatives for the G -conjugacy classes. The associated eigenvalue equation is

$$(6.6) \quad DUv^{(g)} = U(\mathbf{1}_H)(g) \cdot v^{(g)}.$$

- (ii) *Assume the integer k makes the operator $A := UD + kI$ nonsingular, that is, the class function ϕ on G defined by*

$$\phi(g) := U(\mathbf{1}_H)(g) + k$$

is nonvanishing. Then the matrix for the inverse operator A^{-1} has the following (χ, ψ) -entry for any G -irreducibles χ, ψ :

$$\begin{aligned} A_{\chi, \psi}^{-1} &= \left\langle \chi, \frac{\psi}{\phi} \right\rangle_G \\ &= \left\langle \chi \cdot \bar{\psi}, \frac{1}{\phi} \right\rangle_G = \left\langle \frac{\chi}{\phi}, \psi \right\rangle_G = A_{\psi, \chi}^{-1} \end{aligned}$$

where $\langle -, - \rangle_G$ denotes the usual inner product for class functions on G .

- (iii) *Let $s_{|\text{Irr}(G)|}$ be the last Smith normal form entry over \mathbb{Z} for $UD + kI$. Then $s_{|\text{Irr}(G)|} = 0$ when $UD + kI$ is singular, and when $UD + kI$ is nonsingular, $s_{|\text{Irr}(G)|}$, it is the smallest positive integer s for which the class function $s \cdot \frac{1}{\phi}$ is a virtual character (that is, an element of $\mathbf{Cl}_{\mathbb{Z}}(G)$).*

Proof. For (i), see the very simple proof in Okada [17, Appendix, Theorem A].

For (ii), note that $A := UD + kI$ is diagonalizable, with the same complete system of eigenvectors $v^{(g)}$ having eigenvalues $\phi(g) = U\mathbf{1}_H(g) + k$. Since *any* class function α on G satisfies

$$\sum_{\chi} \langle \chi, \alpha \rangle_G \cdot \chi(g) = \alpha(g),$$

one can apply this to the class function $\alpha = \frac{\psi}{\phi}$ where ψ is some G -irreducible character, and conclude that

$$\sum_{\chi} \left\langle \chi, \frac{\psi}{\phi} \right\rangle_G \chi(g) = \frac{\psi(g)}{\phi(g)}.$$

In other words, letting B be the square matrix with rows and columns indexed by G -irreducibles, and having entries

$$B_{\chi, \psi} := \left\langle \chi, \frac{\psi}{\phi} \right\rangle_G,$$

one concludes that, for each g in G ,

$$\sum_{\chi} B_{\chi, \psi} \cdot (v^{(g)})_{\chi} = \phi(g)^{-1} \cdot (v^{(g)})_{\psi}.$$

In other words, $Bv^{(g)} = \phi(g)^{-1}v^{(g)}$. Since this is true for every g in G , one concludes that $B = A^{-1}$.

For (iii), note that Corollary 8.2 and part (ii) above imply that $s_{|\text{Irr}(G)|}$ is the smallest positive integer that makes

$$s \left\langle \chi \cdot \overline{\psi}, \frac{1}{\phi} \right\rangle_G = \left\langle \chi \cdot \overline{\psi}, \frac{s}{\phi} \right\rangle_G$$

an integer for all G -irreducibles χ, ψ . By taking $\psi = \mathbf{1}$, one can see that it is necessary for s to make $\langle \chi, \frac{s}{\phi} \rangle_G$ an integer for all G -irreducibles, that is, $\frac{s}{\phi}$ is a virtual character. However, this is also a sufficient condition on s , since $\chi \cdot \overline{\psi}$ is the character of a genuine (and hence also virtual) G -representation. \square

We wish to apply the previous proposition with $H = G_{n-1}$ and $G = G_n$ in an r -differential tower of groups $1 = G_0 < G_1 < \dots$ as in Definition 6.1. In this case, there is a nice recurrence for the class function $\phi = \frac{1}{U(\mathbf{1}_{n-1})+k}$ on G_n that appears in the proposition, where here $\mathbf{1}_n$ is the trivial character of G_n , and U, D denote induction $\text{Ind}_{G_{n-1}}^{G_n}$ and restriction $\text{Res}_{G_{n-1}}^{G_n}$ of class functions. In particular, transitivity of induction/restriction means that $U^j = \text{Ind}_{G_{n-j}}^{G_n}$ and $D^j = \text{Res}_{G_{n-j}}^{G_n}$ for all j and n .

Proposition 6.10. *In an r -differential tower of groups, one has for every n and every k in \mathbb{C} the following equality among class functions in $\mathbf{Cl}(G_n)$:*

$$\frac{1}{U(\mathbf{1}_{n-1}) + r + k} = \sum_{j=0}^n (-1)^j \frac{U^j(\mathbf{1}_{n-j})}{(j+1)!_{r,k}}.$$

where we recall that $j!_{r,k} := (r+k)(2r+k)(3r+k) \cdots (jr+k)$.

Proof. We begin with two simple facts. First note that (6.1) implies

$$\begin{aligned} D \left(\frac{1}{U(\mathbf{1}_n) + r + k} \right) &= \frac{1}{DU(\mathbf{1}_n) + r + k} \\ (6.7) \quad &= \frac{1}{UD(\mathbf{1}_n) + 2r + k} \\ &= \frac{1}{U(\mathbf{1}_{n-1}) + 2r + k}. \end{aligned}$$

Next, note that if G is a finite group, and H a subgroup, then for any class functions ψ, ϕ on H, G , respectively, one can use Frobenius reciprocity to show

$$(6.8) \quad \text{Ind}_H^G(\psi \cdot \text{Res}_H^G \phi) = (\text{Ind}_H^G \psi) \cdot \phi.$$

We now show the asserted equality by induction on n . In the base case $n = 0$, one can see that both sides are the constant class function $\frac{1}{r+k}$.

In the inductive step, assume that the assertion holds for n , and manipulate the sum on the right side of the proposition for $n + 1$ as follows:

$$\begin{aligned}
\sum_{j=0}^{n+1} (-1)^j \frac{U^j(\mathbf{1}_{n+1-j})}{(j+1)!_{r,k}} &= \frac{1}{r+k} - \sum_{j=1}^{n+1} (-1)^j \frac{U^j(\mathbf{1}_{n+1-j})}{(j+1)!_{r,k}} \\
&\stackrel{(\ell:=j-1)}{=} \frac{1}{r+k} \left(1 - U \sum_{\ell=0}^n (-1)^\ell \frac{U^\ell(\mathbf{1}_{n-\ell})}{(\ell+1)!_{r,k+r}} \right) \\
&\stackrel{(\text{induction})}{=} \frac{1}{r+k} \left(1 - U \left(\frac{1}{U(\mathbf{1}_{n-1}) + 2r+k} \right) \right) \\
&\stackrel{(6.7)}{=} \frac{1}{r+k} \left(1 - U \left(\mathbf{1}_n \cdot D \left(\frac{1}{U(\mathbf{1}_n) + r+k} \right) \right) \right) \\
&\stackrel{(6.8)}{=} \frac{1}{r+k} \left(1 - \frac{U(\mathbf{1}_n)}{U(\mathbf{1}_n) + r+k} \right) \\
&= \frac{1}{r+k} \left(\frac{r+k}{U(\mathbf{1}_n) + r+k} \right) \\
&= \frac{1}{U(\mathbf{1}_n) + r+k}
\end{aligned}$$

finishing the induction. \square

We need one further general fact about r -differential posets that does not involve groups.

Proposition 6.11. *Let P be an r -differential poset P , and $j_{m-1} \triangleleft j_m$ any covering pair in P with the property that j_{m-1}, j_m both cover at most one element of P .*

Then for any integer $n \geq m$ one can extend this to a saturated chain

$$j_{m-1} \triangleleft j_m \triangleleft j_{m+1} \triangleleft \cdots \triangleleft j_{n-1} \triangleleft j_n$$

in which each j_ℓ covers at most one element of P .

Proof. Induct on n , with base case $n = m$ trivial. Since j_n covers exactly one element, it is covered by exactly $r + 1$ distinct elements, say a_1, a_2, \dots, a_{r+1} . We wish to show that some a_i covers only j_n , and hence plays the role of j_{n+1} in extending our chain.

So assume not, that is, there exists elements a_1^-, \dots, a_{r+1}^- all distinct from j_n with the property that a_i covers a_i^- for each i . For each i , since a_i^- and j_n are covered by at least one element in common, namely a_i , they must both cover some element in common, which must be j_{n-1} . Thus $j_n, a_1^-, \dots, a_{r+1}^-$ all cover j_{n-1} , and our inductive hypothesis says that j_{n-1} covers at most one element, so it is covered by at most $r + 1$ elements. This forces $a_i^- = a_j^-$ for some $i \neq j$. But then since $j_n, a_i^- (= a_j^-)$ cover only one element j_{n-1} , they must be covered by a unique element, forcing $a_i = a_j$, a contradiction. \square

Theorem 6.12. *Given an r -differential tower of groups, its r -differential poset has the last Smith entry for $A := DU_n + kI$ equal to 0 when A is singular, and given by the formula*

$$s_{p_n} = \begin{cases} (n+1)!_{r,k} & \text{if } r \geq 2 \\ (n-1)!_{1,k} \cdot (n+1+k) & \text{if } r = 1 \end{cases}$$

when A is nonsingular.

Proof. By Proposition 6.9(iii), it suffices to show that when $DU_n + kI$ is nonsingular, the integers shown on the right side of the theorem are the smallest positive integers s for which the class function $\frac{s}{\phi}$, where $\phi(g) = \text{Ind}_{G_{n-1}}^{G_n}(\mathbf{1}_{n-1}) + k$ is a virtual character of G_n .

By Proposition 6.10, this is same as saying that

$$s \cdot \sum_{j=0}^n (-1)^j \frac{\text{Ind}_{G_{n-j}}^{G_n} \mathbf{1}_{n-j}}{(j+1)!_{r,k}}$$

is a virtual G_n -character. Because $j!_{r,k}$ divides $(j+1)!_{r,k}$, it is clear that $(n+1)!_{r,k}$ is one such positive integer s , for any r . Thus s_{p_n} will divide $(n+1)!_{r,k}$. Also, since when $r = 1$ one has $G_0 = G_1 = 1$, and since

$$\frac{1}{n!_{1,k}} - \frac{1}{(n+1)!_{1,k}} = \frac{1}{(n-1)!_{1,k}(n+1+k)},$$

one finds that $(n-1)!_{1,k}(n+1)_{1,k}$ is one such positive integer s for $r = 1$. Thus s_{p_n} will divide $(n-1)!_{1,k}(n+1+k)$ when $r = 1$.

We wish to show that these divisibilities are actually equalities. First note every irreducible G_n -character appears in the irreducible decomposition of $\text{Ind}_{G_0}^{G_n} \mathbf{1}_0$ since $G_0 = 1$ is the trivial subgroup of G_n . Likewise for $\text{Ind}_{G_1}^{G_n} \mathbf{1}_1$ when $r = 1$ since $G_1 = G_0 = 1$ when $r = 1$. Hence, it suffices to show that there is at least one irreducible G_n -character χ which does not appear in the irreducible decomposition of $\text{Ind}_{G_\ell}^{G_n} \mathbf{1}_\ell$ for $\ell \geq 1$ if $r \geq 2$, and for $\ell \geq 2$ if $r = 1$.

This is provided by Proposition 6.11: find a G_n -irreducible $j_n := \chi$ lying atop a saturated chain $j_0 < j_1 < \dots < j_n$, each of whose elements covers a unique element of P , and which starts

- for $r \geq 2$, in ranks $0, 1$ with $j_0 = \mathbf{1}_0 < j_1 = \psi$ for some nontrivial G_1 -irreducible ψ ,
- for $r = 1$, in ranks $0, 1, 2$ with $j_0 = \mathbf{1}_0 < j_1 = \mathbf{1}_1 < j_2 = \psi$ for the unique nontrivial G_2 -irreducible ψ .

In both cases, one can see that this character χ will have the desired property of not appearing in the appropriate inductions $\text{Ind}_{G_\ell}^{G_n} \mathbf{1}_\ell$. \square

Corollary 6.13. *Given an r -differential tower of groups,*

- (i) *the associated r -differential poset has affirmative answer to Stanley's Question 2.2, that is, its rank numbers p_n satisfy $\Delta p_n > 0$ for $n \geq 2$, and*
- (ii) *for all integers k the last Smith invariant s_{p_n} of $DU_n + kI$ obeys the prediction of Conjecture 1.1: it is zero when $DU_n + kI$ is singular, and equal to the product of the distinct eigenvalues of $DU_n + kI$ when it is nonsingular.*

In particular, these assertions apply to the r -differential Cartesian products \mathbf{Y}^r of Young's lattice.

Proof. For a fixed $n \geq 2$, to show that $\Delta p_n > 0$, apply Theorem 6.12 with the integer k chosen so that $r + k$ is a prime that divides *none* of

$$2r + k, 3r + k, \dots, (n+1)r + k$$

(e.g. pick p to be a prime larger than $(n+1)r$ and take $k = p - r$). Theorem 6.12 tells us that the prime $r + k$ divides the last Smith form entry s_{p_n} for $DU_n + kI$

over \mathbb{Z} , and hence it also divides

$$\det(DU_n + kI) = (r + k)^{\Delta p_n} (2r + k)^{\Delta p_{n-1}} \cdots (nr + k)^{\Delta p_1} ((n + 1)r + k)^{p_0}$$

according to Proposition 2.1. Since $r + k$ is a prime that can only divide the first factor on the right, it must be that $\Delta p_n > 0$.

The last assertion of the corollary is now an immediate consequence of Theorem 6.12, Proposition 2.1, and Proposition 8.4 \square

Remark 6.14. When G is a *nonabelian* finite group, the up and down maps for the towers of wreath products $G_n = \mathfrak{S}_n[G] = G \wr \mathfrak{S}_n$ were also studied by Okada [17]. In this situation, one can see that the branchings are not multiplicity-free, already from $G_1 = G$ down to the trivial group G_0 . Hence one does not have an r -differential tower of groups as we have defined it, and our proof of Corollary 6.13 does not quite apply. However, its conclusions for the up and down maps are still valid, as we now explain.

In this situation, $\text{Irr}(G_n)$ can be naturally indexed by the elements at rank n in \mathbf{Y}^m where $m = |\text{Irr}(G)|$ is the number of complex irreducible G -characters, or the number of conjugacy classes in G . Okada shows [17, Corollary to Proposition 5.1] that one still has $DU_n - UD_n = rI$ in general.

Thus by placing the branching multiplicities on the edges of \mathbf{Y}^m , one obtains the structure of a *self-dual graded graph* in the sense of Fomin [4, 5]. Our results such as Proposition 2.1, 6.9 still apply, with the same proofs. However, in the proof of Theorem 6.12 one needs a substitute for Proposition 6.11, which we currently don't know how to generalize to all self-dual graded graphs.

Nevertheless, for the wreath products G_n , one can easily produce such a substitute as follows. Take any nontrivial G -irreducible $\psi \neq \mathbf{1}_G$ acting on a vector space V . Let χ be the representation of G_n acting on the n -fold tensor product $V \otimes \cdots \otimes V$, in which the subgroup \mathfrak{S}_n of G_n acts by permuting tensor positions and the Cartesian product subgroup G^n acts as by ψ within each component. This representation χ turns out to be a G_n -irreducible, and its restriction to $G_1 (= G)$ is a multiple of ψ . Hence by Frobenius reciprocity, χ is not contained in the irreducible support of $\text{Ind}_{G_1}^{G_n} \mathbf{1}_{G_1}$, as desired.

7. REMARKS ON DUAL GRADED GRAPHS

Our conjectures deal only with differential posets, not dual graded graphs, because it has not been clear to us what the general setting should be.

In particular, Remark 6.14 perhaps suggests that one should consider as an intermediate level of generality the *self-dual graded graphs*. We have not tested this extensively.

When considering dual graded graphs which are not self-dual, one finds that the naive generalizations of Conjectures 1.1, 2.3 and 2.4 fail; we present specific counterexamples below. However, we also present some examples in which Conjecture 2.4 holds for at least one of the two graphs in a pair of dual graded graphs.

Conjecture 2.4 fails already for the up and down maps in the left graph out of the pair of dual graded graphs pictured in Figure 4.

Conjecture 2.3 fails for the dual graded graphs $S\mathbf{Y}$ of *shifted shapes* considered by Fomin [4, 5]: they have rank size p_n equal to the number of partitions of n into distinct parts, for which $\Delta p_4 = 0 < 1 = \Delta p_3$.

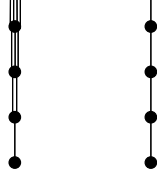
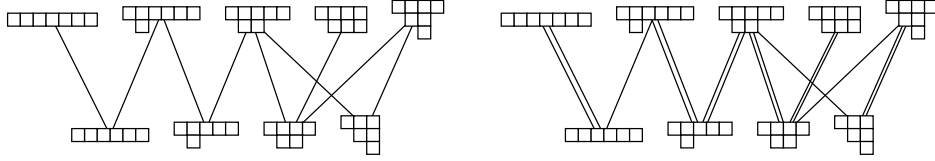


FIGURE 4. Counterexample for Conjecture 2.4

The dual graphs of shifted shapes also provide a counterexample to Conjecture 1.1, when one considers DU_7 .

FIGURE 5. Ranks 6 and 7 of SY .

One finds that

$$DU_7 = \begin{bmatrix} 4 & 2 & 2 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 1 & 2 & 5 & 2 & 0 \\ 0 & 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 2 & \\ & & & & 120 \end{bmatrix},$$

while its eigenvalues 1, 2, 3, 5, 8 would have predicted the diagonal Smith entries as 1, 1, 1, 1, 240.

On the other hand, we list in the next proposition a number of known graded graphs, each a part of a pair of dual graded graphs, having coker U_n free for all n ; see Nzeutchap [15, 16] for definitions and background on these objects.

Proposition 7.1. *In the following graded graphs, coker U_n is free for all n :*

- (1) *the lifted binary tree,*
- (2) *the bracket tree,*
- (3) *Binword,*
- (4) *the lattice of binary trees, and*
- (5) *the lattice of shifted shapes from the pair SY with all edges labelled one.*

Proof. For (1) and (2) this is clear because the graded graphs in question are trees. It therefore remains to check this for (3),(4),(5). For each of these posets P , one can exhibit

- a natural way to linearly order each rank P_n , and
- a $(P-)$ order-preserving injection $P_n \xrightarrow{\phi} P_{n+1}$, so that
- $\phi(x)$ is always the earliest element in the linear order on P_{n+1} which lies above x in P .

From this it follows that U_n has free cokernel, because the sets P_n and $\phi(P_n)$ will index the columns and rows of a $p_n \times p_n$ unitriangular submatrix.

For (3), let $w = w_1 w_2 \cdots w_n$ be a binary word of rank n in Binword, and linearly order them by the usual integer order on the numbers they represent in binary. Then let $\phi(w)$ be obtained from w by inserting a 0 into the second position:

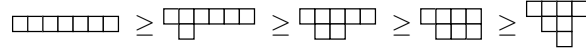
$$\phi(w) = w_1 0 w_2 \cdots w_n.$$

For (4), linearly order each rank in the lattice of binary trees by saying say $T \geq T'$ if the leftmost vertex in which they differ is in T . For example, one has



Given a tree T , let $\phi(T)$ be obtained by adding a left leaf in the leftmost available position.

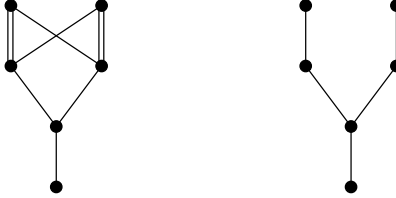
For the lattice of shifted shapes, the linear order on each rank is the natural lexicographic ordering. For example,



Given a shifted shape λ of rank n , let $\phi(\lambda)$ be obtained from it by removing the bottom, rightmost box. \square

Question 7.2. How should the conjectures in this paper generalize to dual graded graphs?

We have noticed that in a large number of examples, the torsion part of the group $\text{coker}(U_n)$ is killed by some power of the product of all edge-multiplicities for the edges between ranks n and $n + 1$. However, this also does not hold in general: the following partial dual graded graphs



have $\text{coker } U_2 \cong \mathbb{Z}/3\mathbb{Z}$, which is not killed by any power of 2 (the only nontrivial edge multiplicity between ranks 2 and 3). One can extend this partial pair to a genuine pair of dual graded graphs by a fairly obvious dual graded graph analogue of Wagner's construction.

8. APPENDIX: ON SMITH AND PARAMETRIZED SMITH FORMS

This appendix discusses Smith forms over rings R which are unique factorization domains (UFD's), but not necessarily principal ideal domains (PID's). See also Kuperberg [9] for other useful facts about this, and interesting examples where such Smith forms exist. Also discussed here is the significance of a square matrix having a *parametrized Smith form*, as defined in the Introduction.

8.1. Smith forms. Let R be a commutative ring. Recall that a matrix A in $R^{m \times n}$ is in *Smith form* if it is diagonal in the sense that its off-diagonal entries vanish, and its diagonal entries $s_i = a_{ii}$ have s_{i+1} divisible by s_i in R for each $i = 1, 2, \dots, \min\{m, n\} - 1$. Recall also that A has a *Smith form over R* if there exists change-of-bases P, Q in $GL_m(R), GL_n(R)$, respectively, such that PAQ is in Smith form.

It is well-known that when R is a PID, every matrix A in $R^{m \times n}$ has a Smith form, but this is not guaranteed for an arbitrary ring R . The following proposition characterizing the diagonal entries in the Smith form is well-known in the case where R is a PID, but holds more generally when R is a UFD.

Proposition 8.1. *Let R be a UFD, and assume that A in $R^{m \times n}$ has a Smith form over R , with diagonal entries s_1, s_2, \dots . Then for each i , the partial product $s_1 s_2 \cdots s_i$ can be interpreted as the greatest common divisor $d_i(A)$ of all determinants of $i \times i$ minors of A .*

Consequently, the entries s_i are determined uniquely up to units in R^\times by the formula

$$(8.1) \quad s_i = \frac{d_i(A)}{d_{i-1}(A)},$$

where $d_0(A)$ is taken to be 1.

Proof. Since greatest common divisors in a UFD are well-defined up to units in R^\times , the second assertion about unique determination follows once one has shown the formula (8.1)

To show formula (8.1), first note that it is trivial when A itself is in Smith form, due to the divisibility relations among the diagonal entries s_i .

Secondly, note that $d_1(PAQ) = d_1(AQ) = d_1(A)$ since invertible row or column operations over R cannot affect the set of common divisors of the entries of a matrix. Since we assumed that one can find P, Q with PAQ in Smith form, the assertion holds for $i = 1$.

For $i \geq 2$, recall that A represents an R -module homomorphism between free R -modules

$$A : R^n \rightarrow R^m.$$

Apply the i^{th} exterior power functor to obtain a matrix $\wedge^i A$ in $R^{\binom{m}{i}} \times R^{\binom{n}{i}}$, representing the R -module homomorphism between the free R -modules

$$\wedge^i A : \wedge^i R^n \rightarrow \wedge^i R^m,$$

with respect to the standard basis of decomposable wedges $e_I := \wedge_{j \in I} e_j$ indexed by i -element subsets I of $\{1, 2, \dots, n\}$ or $\{1, 2, \dots, m\}$ for the source and target. Since the (I, J) entry of $\wedge^i A$ is the determinant of the $i \times i$ minor $A_{I,J}$, one concludes that $d_i(A) = d_1(\wedge^i A)$. As functoriality implies that

$$\wedge^i(PAQ) = \wedge^i P \cdot \wedge^i A \cdot \wedge^i Q,$$

and also implies the invertibility of $\wedge^i P, \wedge^i Q$, one concludes that

$$d_i(PAQ) = d_1(\wedge^i(PAQ)) = d_1(\wedge^i P \cdot \wedge^i A \cdot \wedge^i Q) = d_1(\wedge^i A) = d_i(A).$$

Since it was assumed that one can find P, Q with PAQ in Smith form, the assertion holds for all i . \square

The following corollary characterizing the last Smith form entry is used in Section 6.3.

Corollary 8.2. *Let R be a UFD, with field of fractions K . Assume that a square matrix A in $R^{n \times n}$ has a Smith form over R . Then the last diagonal entry s_n in its Smith form is*

- zero if and only if A is singular, and
- the smallest denominator for A^{-1} in $K^{n \times n}$, in the sense that any s in R for which sA^{-1} lies in $R^{n \times n}$ will have s_n dividing s .

Proof. For the first assertion, note that since A is square and has a Smith form PAQ with diagonal entries s_1, s_2, \dots, s_n , one has

$$s_1 \cdots s_n = \det PAQ = \det P \cdot \det A \cdot \det Q.$$

Note that $\det P, \det Q$ lie in R^\times since P, Q invertible. Hence A is singular if and only if $\det A = 0$ if and only if $s_1 \cdots s_n = 0$ if and only if some $s_i = 0$ if and only if $s_n = 0$.

For the second assertion, recall that

$$(A^{-1})_{ij} = \frac{(-1)^{i+j}}{\det A} \det(A^{ij})$$

where A^{ij} is the $(n-1) \times (n-1)$ minor of A obtained by removing row i and column j . Thus one has

$$\begin{aligned} sA^{-1} \in R^{n \times n} &\Leftrightarrow s \frac{\det A^{ij}}{\det A} \in R \text{ for all } i, j \\ &\Leftrightarrow \det A \mid s \det A^{ij} \text{ for all } i, j \\ &\Leftrightarrow \det A \mid \gcd_{i,j}(s \det A^{ij}) = s \gcd_{i,j}(\det A^{ij}) \\ &\Leftrightarrow s_1 s_2 \cdots s_{n-1} s_n = \det A \mid s d_{n-1}(A) = s \cdot s_1 s_2 \cdots s_{n-1} \\ &\Leftrightarrow s_n \mid s, \end{aligned}$$

where Proposition 8.1 was used in the second-to-last equivalence. \square

Assume R is an integral domain with fraction field K , and let A in $R^{m \times n}$ have rank r when considered as a matrix in $K^{m \times n}$. If A also has a Smith form over R , then in its diagonal entries, the first r of them s_1, s_2, \dots, s_r will be nonzero, and the rest zero. Note that the simplest possible Smith form in this situation, where $s_1, \dots, s_n \in \{0, 1\}$, occurs if and only if the cokernel

$$\text{coker} A := R^m / \text{im}(A : R^n \rightarrow R^m)$$

is free, and isomorphic to R^{m-r} . The following equivalent characterization when rank $r = n$ is useful in the context of Conjecture 2.4.

Proposition 8.3. *Let R be an integral domain. Assume U in $R^{m \times n}$ has rank n over the fraction field K of R , so in particular, $n \leq m$. Then $\text{coker}(U) \cong R^{m-n}$ if and only if $\text{coker}(UU^t) \cong R^{m-n}$.*

Proof. For the forward implication, if $\text{coker}(U) \cong R^{m-n}$ then the short exact sequence

$$0 \rightarrow \text{im} U \rightarrow R^m \rightarrow \text{coker}(U) \rightarrow 0$$

will split, since $\text{coker}(U)$ is free, and hence projective. Therefore

$$R^m = \text{im}(U) \oplus \text{coker}(U)$$

which means that there exist change-of-bases P, Q in $GL_m(R), GL_n(R)$ so that $PUQ = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$. This gives

$$PU = \begin{bmatrix} I_n \\ 0 \end{bmatrix} Q^{-1} = \begin{bmatrix} Q^{-1} \\ 0 \end{bmatrix} = \begin{bmatrix} Q^{-1} & 0 \\ 0 & I_{m-n} \end{bmatrix} \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

Therefore $\hat{P}U = \begin{bmatrix} I_n \\ 0 \end{bmatrix}$ where $\hat{P} := \begin{bmatrix} Q & 0 \\ 0 & I_{m-n} \end{bmatrix} P$ is unimodular. This implies

$$\hat{P}UU^t\hat{P}^t = \begin{bmatrix} I_n \\ 0 \end{bmatrix} [I_n \quad 0] = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence $\text{coker}(UU^t)$ is also free.

For the reverse implication, assume $\text{coker}(UU^t) \cong R^{m-n}$. This implies that $\text{rank}_K UU^t = n$. Note that $\text{im}(UU^t) \subseteq \text{im}U$, which has two implications. Firstly, it implies

$$(n =) \text{rank}_K UU^t \leq \text{rank}_K U$$

and since U has dimensions $m \times n$, this forces

$$\text{rank}_K U = n = \text{rank}_K UU^t.$$

Secondly, it gives a surjection

$$\text{coker}(UU^t) \xrightarrow{\pi} \text{coker}(U)$$

between two R -modules of the same rank. Since this implies that $\ker \pi$ has K -dimension 0, it must be all R -torsion as an R -module, and hence zero because it is a submodule of the free module $\text{coker}(UU^t) \cong R^{m-n}$. Thus π is an isomorphism, and $\text{coker}(U) \cong R^{m-n}$. \square

8.2. Parametrized Smith forms. Recall that a square matrix A in $R^{n \times n}$ is said to have a parametrized Smith form if $A + tI$ in $R[t]^{n \times n}$ has a Smith form over $R[t]$. This has strong consequences.

Proposition 8.4. *Let R be an integral domain, with field of fractions K , and \overline{K} the algebraic closure of K .*

Assume A in $R^{n \times n}$ has a parametrized Smith form for $A + tI$ with the diagonal entries $s_1(t), \dots, s_n(t)$ in $R[t]$.

- (i) *The $s_i(t)$ can all chosen with leading coefficient 1 (monic) in $R[t]$.*
- (ii) *For each λ in \overline{K} , the (geometric) eigenvalue multiplicity*

$$\dim_{\overline{K}} \ker(A - \lambda I) = m$$

can be read off of the $s_i(t)$ as follows: $t + \lambda$ is a factor in $\overline{K}[t]$ of the last m entries

$$s_n(t), s_{n-1}(t), \dots, s_{n-(m-1)}(t)$$

but no other $s_i(t)$.

- (iii) *A is semisimple (=diagonalizable in $\overline{K}^{n \times n}$) if and only the $s_i(t)$ have no repeated factors when split completely in $\overline{K}[t]$.*

(iv) In the situation where A is semisimple, the $s_i(t)$ factor as follows:

$$s_{n+1-i}(t) = \prod_{\substack{\lambda \in \overline{K}: \\ \dim_{\overline{K}} \ker(A - \lambda I) \geq i}} (t + \lambda).$$

So, for example, the last parametrized Smith entry factors as

$$s_n(t) = \prod_{\text{eigenvalues } \lambda \in \overline{K} \text{ for } A} (t + \lambda).$$

Proof. By assumption, there exist $P(t), Q(t)$ lying in $GL_n(R[t])$ for which

$$(8.2) \quad P(t)(A + tI)Q(t) = \text{diag}(s_1(t), \dots, s_n(t))$$

where here $\text{diag}(a_1, \dots, a_n)$ denotes the $n \times n$ square diagonal matrix with eigenvalues a_1, \dots, a_n . Since R is an integral domain, the units of $R[t]$ are exactly the units R^\times , that is, independent of t , and hence $\det P(t), \det Q(t)$ both lie in R^\times . Thus one has

$$(8.3) \quad \det P(t) \det(A + tI) \det Q(t) = s_1(t) \cdots s_n(t)$$

where $\det P(t), \det Q(t)$ lie in R^\times .

Assertion (i) now follows: up to scalars in R^\times , the leading terms of the $s_i(t)$ multiply to give the leading term of $\det(A + tI)$, which is 1. Hence these leading terms are all in R^\times , and can be scaled away.

For assertion (ii), note that for any λ in \overline{K} , one can set $t = -\lambda$ in (8.2) to obtain

$$(8.4) \quad P(-\lambda)(A - \lambda I)Q(-\lambda) = \text{diag}(s_1(-\lambda), \dots, s_n(-\lambda))$$

where both $P(-\lambda), Q(-\lambda)$ lie in $GL_n(\overline{K})$ since their determinants are independent of λ and lie in $R^\times (\subset K^\times \subset \overline{K}^\times)$. Consequently, one has

$$m = \dim_{\overline{K}} \ker(A - \lambda I) = \dim_{\overline{K}} \ker \text{diag}(s_1(-\lambda), \dots, s_n(-\lambda))$$

if and only if $t = -\lambda$ is a root of exactly m of the polynomials $s_i(t)$, that is, if and only if $t + \lambda$ is a factor in $\overline{K}[t]$ of $s_n(t), s_{n-1}(t), \dots, s_{n-(m-1)}(t)$ and no others.

For assertions (iii) and (iv), recall that one has A semisimple if and only if for each λ the (geometric) multiplicity $m = \dim_{\overline{K}} \ker(A - \lambda I)$ is the same as the multiplicity of $t = -\lambda$ as a root of $\det(A + tI)$. We claim that the latter is the same as its multiplicity as a root of $s_1(t) \cdots s_n(t)$, due to (8.3). Since we know $t = -\lambda$ is always a root of the last m factors in this product and no others, it is a root of multiplicity m if and only if it is a simple root for each of these last m factors. This gives all the remaining assertions. \square

We illustrate, by way of examples, that having A in $R^{n \times n}$ with a parametrized Smith form for $A + tI$ in $R[t]$ implies neither that A is semisimple, nor that its eigenvalues lie in R or K .

Example 8.5. Let $R = \mathbb{Z}$ and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which is not semisimple. Nevertheless, one can bring $A + tI$ into Smith form over $\mathbb{Z}[t]$ by these row and column operations:

$$A + tI = \begin{bmatrix} t & 1 \\ 0 & t \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ -t^2 & t \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ -t^2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & t^2 \end{bmatrix}.$$

Note that one of the diagonal entries $s_2(t) = t^2$ has a double root in this case, as predicted by Proposition 8.4 above.

Example 8.6. Let $R = K = \mathbb{Q}$ and

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

which is semisimple, but with eigenvalues $\pm i$ that do not lie in $R = K$. However, since R is a field and hence $R[t]$ is a PID, one can certainly bring $A + tI$ into Smith form over $\mathbb{Q}[t]$ by these row and column operations:

$$A + tI = \begin{bmatrix} t & 1 \\ -1 & t \end{bmatrix} \sim \begin{bmatrix} 0 & t^2 + 1 \\ -1 & t \end{bmatrix} \sim \begin{bmatrix} 0 & t^2 + 1 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & t^2 + 1 \end{bmatrix}.$$

Note that each of the eigenvalues is at most a simple root of the last Smith entry $s_2(t) = t^2 + 1 = (t + i)(t - i)$.

Remark 8.7. Note that if R is itself a PID, then for every value k in R , the matrix $A + kI$ in $R^{n \times n}$ will have a Smith form over R . Thus whenever $A + tI$ also has a Smith form in $R[t]$, it will predict these Smith forms over R by setting $t = k$.

Note also that for any integral domain R , since its field of fractions K always has $K[t]$ a PID, a Smith form for $A + tI$ in $K[t]$ is guaranteed to exist. Thus whenever $A + tI$ also has a Smith form in $R[t]$, it will predict the Smith form in $K[t]$. However, knowledge of the eigenvalues of A in \overline{K} will already predict this Smith form for $A + tI$ in $K[t]$, by the argument in Proposition 8.4.

Remark 8.8. To give some sense of how rarely one finds a parametrized Smith form for $A + tI$ when R is not a field and A lies in $R^{n \times n}$, we note its scarcity among *diagonal matrices*.

Proposition 8.9. *Let $A = \text{diag}(\lambda_1, \dots, \lambda_n)$ be a diagonal matrix in $R^{n \times n}$. If $A + tI$ has a Smith form in $R[t]$ then every nonzero difference $\lambda_i - \lambda_j$ is a unit in R^\times .*

In particular, for $R = \mathbb{Z}$, a diagonal matrix A has a parametrized Smith form for $A + tI$ in $\mathbb{Z}[t]$ if and only if the diagonal entries take on at most two values in \mathbb{Z} , and those two values differ by 1.

Proof. Let λ be among the diagonal entries $\{\lambda_1, \dots, \lambda_n\}$. We will show that $\lambda_i - \lambda$ lies in R^\times for all i using downward induction on the multiplicity m with which λ occurs in the list. Reindex so that λ occurs as the last m diagonal entries in A . Let $P(t)(A + tI)Q(t) = \text{diag}(s_1(t), \dots, s_n(t))$, where we know from Proposition 8.4 how to factor $s_i(t)$ over R . Setting $t = -\lambda$ one obtains

$$\begin{aligned} & P(-\lambda) \text{diag}(\lambda_1 - \lambda, \dots, \lambda_{n-m} - \lambda, \underbrace{0, 0, \dots, 0}_{m \text{ times}}) Q(-\lambda) \\ &= \text{diag}(s_1(-\lambda), \dots, s_{n-m}(-\lambda), \underbrace{0, 0, \dots, 0}_{m \text{ times}}). \end{aligned}$$

However the exact form of each of the nonzero terms $s_i(-\lambda)$ on the right shows that it is a product of factors of the form $\lambda_j - \lambda$ where λ_j has larger multiplicity than m . Thus by induction, each such factor $\lambda_j - \lambda$ is a unit, and hence so is $s_i(-\lambda)$.

Taking cokernels of both sides now gives an R -module isomorphism

$$R^m \oplus \bigoplus_{i=1}^{n-m} R/(\lambda_i - \lambda) \cong R^m.$$

This forces each $R/(\lambda_i - \lambda) = 0$ (and hence $\lambda_i - \lambda$ is forced to be a unit): any x in $R/(\lambda_i - \lambda)$ gives rise to an element $(x, 0, 0, \dots)$ killed by $\lambda_i - \lambda$ on the left, but since only 0 is killed by it in R^m on the right, this forces $x = 0$.

The assertion for $R = \mathbb{Z}$ then follows easily from the fact that $\mathbb{Z}^\times = \{\pm 1\}$, and the difference $+1 - (-1) = 2$ is *not* in \mathbb{Z}^\times . \square

Example 8.10. The following examples illustrate some of the previous remarks.

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$

in $\mathbb{Z}^{2 \times 2}$ has a parametrized Smith form, as

$$A + tI = \begin{bmatrix} t-1 & 0 \\ 0 & t \end{bmatrix} \sim \begin{bmatrix} t-1 & 0 \\ t & t \end{bmatrix} \sim \begin{bmatrix} -1 & -t \\ t & t \end{bmatrix} \sim \begin{bmatrix} -1 & -t \\ 0 & t-t^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & t(t-1) \end{bmatrix}$$

Consequently, it has the same Smith form over $\mathbb{Q}[t]$, and for any integer k , the matrix $A + kI$ will have Smith form over \mathbb{Z} given by setting $t = k$. In particular, the cokernel $\mathbb{Z}^2/\text{im}(A + kI)$ is always isomorphic to the cyclic group $\mathbb{Z}/k(k-1)\mathbb{Z}$, interpreting this as $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$ when $k = 0, 1$.

On the other hand, the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

has *no* Smith form in $\mathbb{Z}[t]$ for

$$A + tI = \begin{bmatrix} t-1 & 0 \\ 0 & t+1 \end{bmatrix}$$

according to Proposition 8.9. The Smith form for $A + tI$ in $\mathbb{Q}[t]$ has diagonal entries $1, (t-1)(t+1)$, but requires dividing by 2 in order to put it into this form. One also sees a wider range of behaviors for the cokernel of $A + kI$ over \mathbb{Z} as the integer k varies:

$$\text{coker}(A + kI) \cong \begin{cases} \mathbb{Z}/(k-1)\mathbb{Z} \oplus \mathbb{Z}/(k+1)\mathbb{Z} & \text{if } k \text{ is odd,} \\ \mathbb{Z}/(k-1)(k+1)\mathbb{Z} & \text{if } k \text{ is even.} \end{cases}$$

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E-mail address: mill1966@math.umn.edu

E-mail address: reiner@math.umn.edu

SCHOOL OF MATHEMATICS, UNIVERSITY OF MINNESOTA, MINNEAPOLIS, MN 55455